

Kreĭn C^* -modules

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Abstract

We introduce a notion of Kreĭn C^* -module over a C^* -algebra and more generally over a Kreĭn C^* -algebra. Some properties of Kreĭn C^* -modules and their categories are investigated.

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1 Introduction

Vector spaces with an indefinite inner product started to appear in physics with the work on special relativistic space-time by H.Minkowski [M] and were later used for the first time in quantum field theory by P.Dirac [D] and W.Pauli [P], but their first mathematical discussion was provided by L.Pontrjagin [Po] and since then they have been an object of study mainly of the Russian school.

Kreĭn spaces, i.e. complete vector spaces equipped with an indefinite inner product, were formally defined by Ju.Ginzburg [Gi] and in their present form by E.Scheibe [Sc]. Their properties have been investigated by several mathematicians such as I.Iohvidov, H.Langer, R.Phillips, M.Naĭmark, M.Kreĭn

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and his school and have been extensively used in quantum field theory via the Gupta-Bleuler [B, G] formalism in quantum electrodynamics.¹ They have been reconsidered in quantum field theory by K.Kawamura [K1, K2] who also proposed axioms for Kreĭn C^* -algebras (involutive algebras of bounded linear operators on a Kreĭn space). Kreĭn spaces also appeared prominently in the definition of semi-Riemannian spectral triples in non-commutative geometry, by A.Strohmaier [Str], M.Paschke, A.Sitarz [PS] and more recently by K.van den Dungen, M.Paschke, A.Rennie [DPR].

Hilbert C^* -modules (complete modules over a C^* -algebra with a C^* -algebra-valued positive inner product) are a generalization of Hilbert spaces where the field of complex numbers is substituted by a general C^* -algebra. They were first introduced in 1953 by I.Kaplanski [K] in the case of commutative unital C^* -algebras. Between 1972 and 1974 W.Paschke [Pa1, Pa2] and M.Rieffel [R1, R2, R3] extended the theory to the case of modules over arbitrary C^* -algebras and after that the subject grew and spread rapidly.

The purpose of this paper is to introduce an extremely elementary notion of Kreĭn C^* -module over a Kreĭn C^* -algebra, generalizing to the module case the usual decomposability condition of a Kreĭn space into its “positive” and “negative” subspaces, and closely following the definition of Kreĭn C^* -module over a C^* -algebra elaborated in S.Kaewumpai [Ka].

In practice, such “decomposable” kind of Kreĭn modules admit a non-canonical splitting as direct sums of Hilbert C^* -modules, with opposite signature that, when we allow the algebra to be a Kreĭn C^* -algebra, can be chosen to be “compatible” with one of the fundamental symmetry automorphisms of the algebra.

We first recall in section 2 (a variant of) the definition of Kreĭn C^* -algebra by K.Kawamura [K1, K2] and, in section 3, for the benefit of the reader, we reproduce in some detail the key definitions and proofs of the main results on Kreĭn C^* -modules over C^* -algebras that were developed in S.Kaewumpai’s thesis [Ka].

In section 4, we further extend the previous definition to cover the case of Kreĭn C^* -modules over Kreĭn C^* -algebras. In the subsequent section 5 we expand the notion of tensor product of Kreĭn C^* -modules over C^* -algebras, formulated in R.Tanadkithirun [T] in order to cover our more general situation and we discuss some of the properties of the categories of modules and bimodules so obtained.

Several examples illustrating the scope of the definitions are presented. Of particular interest are the possible applications to the spectral geometry of semi-Riemannian manifolds and their non-commutative counterparts.

As it can be clearly appreciated by these geometric examples of modules of vector fields over the Clifford algebra of a semi-Riemannian manifold, the notion of Kreĭn C^* -module that is contained here is very specific and corresponds to the special case of tangent bundles admitting a global decomposition as Whitney orthogonal sums of positive and negative definite Hermitian vector sub-bundles i.e. semi-Riemannian manifolds that are time-orientable and space-orientable.² More general notions of Kreĭn C^* -modules are necessary to deal with cases where such global “splitting” is not available, but for now we do not enter this interesting discussion that will very likely also require modifications in the definition of Kreĭn C^* -algebra.

2 Kreĭn C^* -Algebras

The following is a variation of the definition of Kreĭn C^* -algebra introduced by K.Kawamura [K1].

Definition 2.1. A *Kreĭn C^* -algebra* is an involutive complete complex topological algebra (i.e. a complete complex topological vector space with a bilinear continuous product and a continuous involution) \mathcal{A} that admits at least one **fundamental symmetry** i.e. an involutive automorphism

¹See as main references: J.Bognar[Bo], T.Azizov, I.Iohvidov [AI], M.Dritschel, J.Rovnyak [DR], E.Kissin, V.Shulman [KS].

²For details on semi-Riemannian geometry we refer to B.O’Neill [O].

$\alpha : \mathcal{A} \rightarrow \mathcal{A}$ with $\alpha \circ \alpha = \iota_{\mathcal{A}}$ and one Banach algebra norm $\|\cdot\|_{\alpha}$ (inducing the given topology) such that $\|\alpha(a^*)a\|_{\alpha} = \|a\|_{\alpha}^2$, for all $a \in \mathcal{A}$.

Proposition 2.2 (K. Kawamura, Example 2.4, Section 2.3). *The set $\mathcal{B}(K)$ of linear continuous operators on a Kreĭn space K is a Kreĭn C^* -algebra. Every fundamental symmetry J of a Kreĭn space K is associated to a fundamental symmetry $a \mapsto JaJ$, $a \in \mathcal{B}(K)$ of the Kreĭn C^* -algebra $\mathcal{B}(K)$.*

Remark 2.3. Note that although, contrary to K.Kawamura, we assume the existence of a given topology, we do not fix a priori any Banach norm on the Kreĭn C^* -algebra, so that several topologically equivalent Banach norms can exist. Specifically, for every fundamental symmetry α there is a unique norm $\|\cdot\|_{\alpha}$ making \mathcal{A} a C^* -algebra, denoted by \mathcal{A}^{α} , that coincides with \mathcal{A} as a complex algebra and whose involution is given by $x^{\dagger\alpha} := \alpha(x^*)$, for all $x \in \mathcal{A}$.

For example, different fundamental symmetries of a Kreĭn space K , induce operator norms on the Kreĭn C^* -algebra $\mathcal{B}(K)$ of bounded linear operators on K that do not coincide, although they are topologically equivalent. \lrcorner

In the subsequent sections, we will often use the notation \mathcal{A}_+ for the **even part** of the Kreĭn C^* -algebra \mathcal{A} under a fundamental symmetry α , i.e. the C^* -algebra of elements such that $\alpha(x) = x$; and similarly the notation \mathcal{A}_- for the **odd part** of the Kreĭn C^* -algebra \mathcal{A} under a fundamental symmetry α i.e. the Hilbert C^* -module, over \mathcal{A}_+ , of elements such that $\alpha(x) = -x$.

Later on we will see natural situations motivated from semi-Riemannian geometry that seem to require further generalization of the definition of Kreĭn C^* -algebra, but for this work we will mostly limit our consideration to the definition above.

3 Kreĭn C^* -modules over C^* -algebras

In this section, we recall some basic material on unital Kreĭn C^* -modules over unital C^* -algebras that was developed in S.Kaewumpai's Master thesis [Ka]. The material naturally covers, as a special case, the situation of Kreĭn spaces (that are Kreĭn C^* -modules over the C^* -algebra \mathbb{C}) and will be further generalized in the subsequent section where we will consider modules over Kreĭn C^* -algebras.

Recall that, given a unital right module $\mathcal{E}_{\mathcal{A}}$ over a unital C^* -algebra \mathcal{A} , an \mathcal{A} -valued Hermitian **inner product** is a map $\langle \cdot | \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ such that, for all $x, y, z \in \mathcal{E}$, $a \in \mathcal{A}$:

$$\langle z | x + y \rangle = \langle z | x \rangle + \langle z | y \rangle, \quad \langle z | xa \rangle = \langle z | x \rangle a, \quad \langle x | y \rangle^* = \langle y | x \rangle.$$

In the case of unital left modules ${}_{\mathcal{A}}\mathcal{E}$, the second property above is substituted with $\langle ax | z \rangle = a \langle x | z \rangle$. Whenever confusion might arise, we will denote an inner product on $\mathcal{E}_{\mathcal{A}}$ by $\langle \cdot | \cdot \rangle_{\mathcal{E}}$ and an inner product on ${}_{\mathcal{A}}\mathcal{E}$ by ${}_{\mathcal{E}}\langle \cdot | \cdot \rangle$. The direct sum $\mathcal{E} \oplus \mathcal{F}$ of two right (left) unital modules, over the unital C^* -algebra \mathcal{A} , equipped with the inner product defined by $\langle x_1 \oplus y_1 | x_2 \oplus y_2 \rangle_{\mathcal{E} \oplus \mathcal{F}} := \langle x_1 | x_2 \rangle_{\mathcal{E}} + \langle y_1 | y_2 \rangle_{\mathcal{F}}$, for all $x_1, x_2 \in \mathcal{E}$ and $y_1, y_2 \in \mathcal{F}$, is a right (left) unital module over \mathcal{A} called **orthogonal direct sum** of $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}}$.

A Hermitian inner product is **positive** if $\langle x | x \rangle \in \mathcal{A}_{>} := \{a^*a \mid a \in \mathcal{A}\}$, for all $x \in \mathcal{E}$, where $\mathcal{A}_{>}$ denotes the positive part of the unital C^* -algebra \mathcal{A} . An inner product is **non-degenerate** if $\langle x | x \rangle = 0 \Rightarrow x = 0_{\mathcal{A}}$. A unital right (left) **Hilbert C^* -module** \mathcal{E} over the unital C^* -algebra \mathcal{A} is a unital right (left) module on the unital C^* -algebra \mathcal{A} , that is equipped with an \mathcal{A} -valued positive non-degenerated inner product, making it a complete metric space with respect to the norm $\|x\| := \sqrt{\|\langle x | x \rangle\|_{\mathcal{A}}}$.

The family $\mathcal{L}(\mathcal{E}_{\mathcal{A}})$ of \mathcal{A} -linear operators on the right (left) unital \mathcal{A} -module $\mathcal{E}_{\mathcal{A}}$ is a complex algebra with multiplication given by composition of maps. If the modules $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{F}_{\mathcal{A}}$ are equipped with \mathcal{A} -valued inner products, we say that a map $T : \mathcal{E} \rightarrow \mathcal{F}$ is **adjointable** if there exists another map

$S : \mathcal{F} \rightarrow \mathcal{E}$ such that $\langle y \mid T(x) \rangle_{\mathcal{F}} = \langle S(y) \mid x \rangle_{\mathcal{E}}$, for all $x \in \mathcal{E}$ and $y \in \mathcal{F}$. If the inner product is non-degenerate, adjointable maps are necessarily unique and \mathcal{A} -linear and, denoting by T^* the unique adjoint of T we have, for Hermitian inner products, $(T^*)^* = T$, $(T \circ S)^* = S^* \circ T^*$ and $(T_1 + \alpha T_2)^* = T_1^* + \overline{\alpha} T_2^*$ so that the family $\mathcal{B}(\mathcal{E}_{\mathcal{A}})$ of adjointable operators is an involutive complex subalgebra of $\mathcal{L}(\mathcal{E}_{\mathcal{A}})$. In the case of Hilbert C^* -modules, adjointable maps are always continuous and the algebra $\mathcal{B}(\mathcal{E}_{\mathcal{A}})$ is always a unital C^* -algebra when equipped with the operator norm.³

Definition 3.1. If $\mathcal{E}_{\mathcal{A}}$ is module with an inner product $\langle \cdot \mid \cdot \rangle$ over the C^* -algebra \mathcal{A} , its **antimodule**, here denoted by $-\mathcal{E}_{\mathcal{A}}$, is the \mathcal{A} -module $\mathcal{E}_{\mathcal{A}}$ equipped with the opposite inner product. $-\langle \cdot \mid \cdot \rangle$.

Definition 3.2. A unital right **Kreĭn C^* -module** $\mathcal{K}_{\mathcal{A}}$ over the unital C^* -algebra \mathcal{A} is a unital right module over \mathcal{A} with an \mathcal{A} -valued inner product such that $\mathcal{K}_{\mathcal{A}}$ is isomorphic to the “indefinite” orthogonal direct sum $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{+\mathcal{A}} \oplus \mathcal{K}_{-\mathcal{A}}$ of a Hilbert C^* -module $\mathcal{K}_{+\mathcal{A}}$, with the antimodule $\mathcal{K}_{-\mathcal{A}}$ of a Hilbert C^* -module $-\mathcal{K}_{-\mathcal{A}}$. Unital left Kreĭn C^* -modules are similarly defined.

Any such decomposition of a Kreĭn C^* -module over \mathcal{A} will be called a **fundamental decomposition**. Fundamental decompositions are in general not unique.

Definition 3.3. To every fundamental decomposition $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{+\mathcal{A}} \oplus \mathcal{K}_{-\mathcal{A}}$ of a right unital Kreĭn C^* -module $\mathcal{K}_{\mathcal{A}}$ over the unital C^* -algebra \mathcal{A} there is an associated **fundamental symmetry** operator $J : \mathcal{K}_{\mathcal{A}} \rightarrow \mathcal{K}_{\mathcal{A}}$ given by

$$J(x) := x_+ - x_-, \quad \text{where } x = x_+ + x_-$$

is the direct sum decomposition of $x \in \mathcal{K}_{\mathcal{A}}$; and there is an associated Hilbert C^* -module

$$|\mathcal{K}|_{\mathcal{A}} := \mathcal{K}_{+\mathcal{A}} \oplus (-\mathcal{K}_{-\mathcal{A}}).$$

Making use of the unicity of the expression of vectors as sums of components belonging to the direct summands of a fundamental decomposition and of the definition of the fundamental symmetry associated to such decomposition, we obtain the following statement.

Proposition 3.4. Let $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{+\mathcal{A}} \oplus \mathcal{K}_{-\mathcal{A}}$ be a fundamental decomposition of a Kreĭn \mathcal{A} - C^* -module $\mathcal{K}_{\mathcal{A}}$, its fundamental symmetry operator $J : \mathcal{K}_{\mathcal{A}} \rightarrow \mathcal{K}_{\mathcal{A}}$ satisfies the following properties for all $x, y \in \mathcal{K}$ and $a \in \mathcal{A}$:

$$\begin{aligned} J(x+y) &= J(x) + J(y), & J(xa) &= J(x)a, & J \circ J &= \text{Id}_{\mathcal{K}}, \\ \langle J(x) \mid y \rangle_{\mathcal{K}} &= \langle x \mid J(y) \rangle_{\mathcal{K}}, & \text{or equivalently } \langle J(x) \mid J(y) \rangle_{\mathcal{K}} &= \langle x \mid y \rangle_{\mathcal{K}}, \\ \pm \langle (\text{Id}_{\mathcal{K}} \pm J)(x) \mid (\text{Id}_{\mathcal{K}} \pm J)(x) \rangle_{\mathcal{K}} &\geq 0. \end{aligned}$$

Every map $J : \mathcal{K} \rightarrow \mathcal{K}$ that satisfies the previous list of properties is the fundamental symmetry of a unique fundamental decomposition, denoted by $\mathcal{K} = \mathcal{K}_+^J \oplus \mathcal{K}_-^J$, where $\mathcal{K}_{\pm}^J := \{x \in \mathcal{K} \mid J(x) = \pm x\}$. The operators $(\text{Id}_{\mathcal{K}} \pm J)/2$ are a pair of orthogonal projections onto \mathcal{K}_{\pm}^J . The inner product in the Kreĭn C^* -module $\mathcal{K}_{\mathcal{A}} = \mathcal{K}_{+\mathcal{A}}^J \oplus \mathcal{K}_{-\mathcal{A}}^J$ and the inner product in the associated Hilbert C^* -module $|\mathcal{K}|_{\mathcal{A}}^J := \mathcal{K}_{+\mathcal{A}}^J \oplus (-\mathcal{K}_{-\mathcal{A}}^J)$ are related by

$$\langle x \mid y \rangle_{\mathcal{K}} = \langle J(x) \mid y \rangle_{|\mathcal{K}|^J}, \quad \langle J(x) \mid y \rangle_{\mathcal{K}} = \langle x \mid y \rangle_{|\mathcal{K}|^J}, \quad \forall x, y \in \mathcal{K}.$$

Theorem 3.5. If $\mathcal{K}_{\mathcal{A}}$ is a unital right (left) Kreĭn C^* -module over the unital C^* -algebra \mathcal{A} and J_1, J_2 are two fundamental symmetries of \mathcal{K} with associated fundamental decompositions $\mathcal{K} = \mathcal{K}_+^{J_1} \oplus \mathcal{K}_-^{J_1}$

³For all the details of these standard results on Hilbert C^* -modules, the reader can consult N.Landsman [L, L1], E.Lance [La], N.Wegge-Olsen [WO] or S.Kaewunpai [Ka].

and $\mathcal{K} = \mathcal{K}_+^{J_2} \oplus \mathcal{K}_-^{J_2}$, there are two \mathcal{A} -linear continuous bijective maps of Hilbert \mathcal{A} - C^* -modules $T_+^{J_2 J_1} : \mathcal{K}_+^{J_1} \rightarrow \mathcal{K}_+^{J_2}$ and $T_-^{J_2 J_1} : -(\mathcal{K}_-^{J_1}) \rightarrow -(\mathcal{K}_-^{J_2})$, given by $T_+^{J_2 J_1}(x_+^{J_1}) := x_+^{J_2}$ and $T_-^{J_2 J_1}(x_-^{J_1}) := x_-^{J_2}$, where $x = x_+^{J_1} + x_-^{J_1}$ and $x = x_+^{J_2} + x_-^{J_2}$ are the direct sum decompositions of $x \in \mathcal{K}_{\mathcal{A}}$ in each of the fundamental decompositions.

Proof. The map $T_+^{J_2 J_1} : \mathcal{K}_+^{J_1} \rightarrow \mathcal{K}_+^{J_2}$ is adjointable with adjoint given by the map

$$T_+^{J_1 J_2} : \mathcal{K}_+^{J_2} \rightarrow \mathcal{K}_+^{J_1}, \quad \text{defined by} \quad T_+^{J_1 J_2}(y) := y_+^{J_1}, \quad \text{for } y \in \mathcal{K}_+^{J_2},$$

since $\langle T_+^{J_2 J_1}(x) \mid y \rangle_{\mathcal{K}_+^{J_2}} = \langle x_+^{J_2} \mid y_+^{J_2} \rangle_{\mathcal{K}_+^{J_2}} = \langle x \mid y \rangle_{\mathcal{K}} = \langle x_+^{J_1} \mid y \rangle_{\mathcal{K}_+^{J_1}} = \langle x \mid T_+^{J_1 J_2}(y) \rangle_{\mathcal{K}_+^{J_1}}$, for all $x \in \mathcal{K}_+^{J_1}$ and all $y \in \mathcal{K}_+^{J_2}$. In exactly the same way we have that $T_-^{J_2 J_1} : \mathcal{K}_-^{J_1} \rightarrow \mathcal{K}_-^{J_2}$ is adjointable with adjoint given by the map $T_-^{J_1 J_2} : \mathcal{K}_-^{J_2} \rightarrow \mathcal{K}_-^{J_1}$ defined by $T_-^{J_1 J_2}(y) := y_-^{J_1}$, for all $y \in \mathcal{K}_-^{J_2}$. Since an adjointable operator between Hilbert C^* -modules (and also between anti-Hilbert C^* -modules) is necessarily continuous, both $T_+^{J_2 J_1}, T_-^{J_2 J_1}$ and their adjoints $T_+^{J_1 J_2}, T_-^{J_1 J_2}$ are continuous \mathcal{A} -linear maps. If $x, y \in \mathcal{K}_+^{J_1}$ and $T_+^{J_2 J_1}(x) = T_+^{J_2 J_1}(y)$, we have $x_+^{J_1} - y_+^{J_1} = x - y = x_-^{J_2} - y_-^{J_2}$ and since we have $0 \leq \langle x - y \mid x - y \rangle_{\mathcal{K}_+^{J_1}} = \langle x - y \mid x - y \rangle_{\mathcal{K}} = \langle x_-^{J_2} - y_-^{J_2} \mid x_-^{J_2} - y_-^{J_2} \rangle_{\mathcal{K}_-^{J_2}} \leq 0$, via the non-degeneracy of the Kreĭn C^* -module $\mathcal{K}_{\mathcal{A}}$, we obtain the injectivity of $T_+^{J_2 J_1}$.

Let (x_n) be a sequence in $\mathcal{K}_+^{J_1}$ such that $T_+^{J_2 J_1}(x_n)$ converges to a given point $z \in \overline{\text{Im}(T_+^{J_2 J_1})} \subset \mathcal{K}_+^{J_2}$. Since $\langle (x_n - x_m)_-^{J_2} \mid (x_n - x_m)_-^{J_2} \rangle_{\mathcal{K}} \leq 0$, we have

$$\begin{aligned} \|x_n - x_m\|_{\mathcal{K}_+^{J_1}}^2 &= \langle x_n - x_m \mid x_n - x_m \rangle_{\mathcal{K}} \\ &\leq \langle T_+^{J_2 J_1}(x_n - x_m) \mid T_+^{J_2 J_1}(x_n - x_m) \rangle_{\mathcal{K}} = \|T_+^{J_2 J_1}(x_n) - T_+^{J_2 J_1}(x_m)\|_{\mathcal{K}_+^{J_2}}^2, \end{aligned}$$

so that (x_n) , being a Cauchy sequence in the Hilbert C^* -module $\mathcal{K}_+^{J_1}$, converges to a point $x_o \in \mathcal{K}_+^{J_1}$. By continuity of $T_+^{J_2 J_1}$ we get $z = \lim_{n \rightarrow \infty} T_+^{J_2 J_1}(x_n) = T_+^{J_2 J_1}(x_o) \in \text{Im}(T_+^{J_2 J_1})$, that provides the closure of the range of $T_+^{J_2 J_1}$.

The fact that $T_+^{J_2 J_1} : \mathcal{K}_+^{J_1} \rightarrow \mathcal{K}_+^{J_2}$ is an adjointable map between Hilbert C^* -modules with closed range is equivalent (see for example N.Wegge-Olsen [WO, corollary 15.3.9]) to the complementability of the submodule $\text{Im}(T_+^{J_2 J_1}) \subset \mathcal{K}_+^{J_2}$.

If $y \in \text{Im}(T_+^{J_2 J_1})^\perp \subset \mathcal{K}_+^{J_2}$, for all $x \in \mathcal{K}_+^{J_1}$, we get $\langle T_+^{J_1 J_2}(y) \mid x \rangle_{\mathcal{K}_+^{J_1}} = \langle y \mid T_+^{J_2 J_1}(x) \rangle_{\mathcal{K}_+^{J_2}} = 0$ and hence $y \in \text{Ker}(T_+^{J_1 J_2})$. Since $T_+^{J_1 J_2}$ is injective, we have $\text{Im}(T_+^{J_2 J_1})^\perp = \{0\}$ and since $\text{Im}(T_+^{J_2 J_1})$ is complementable, from $\mathcal{K}_+^{J_2} = \text{Im}(T_+^{J_2 J_1}) \oplus \text{Im}(T_+^{J_2 J_1})^\perp = \text{Im}(T_+^{J_2 J_1})$, we obtain the surjectivity of $T_+^{J_2 J_1}$. \square

Theorem 3.6. *If $\mathcal{K}_{\mathcal{A}}$ is a unital right (left) Kreĭn C^* -module, over \mathcal{A} , with two fundamental symmetries J_1 and J_2 with associated direct sum decompositions $\mathcal{K} = \mathcal{K}_+^{J_1} \oplus \mathcal{K}_-^{J_1} = \mathcal{K}_+^{J_2} \oplus \mathcal{K}_-^{J_2}$, then the Hilbert C^* -modules $|\mathcal{K}|^{J_1} := \mathcal{K}_+^{J_1} \oplus \mathcal{K}_-^{J_1}$ and $|\mathcal{K}|^{J_2} := \mathcal{K}_+^{J_2} \oplus \mathcal{K}_-^{J_2}$ have equivalent norms. Hence on $\mathcal{K}_{\mathcal{A}}$ there is a natural topology called the **strong topology**.*

Proof. By theorem 3.5, we have two bijective \mathcal{A} -linear adjointable (hence continuous) maps

$$T_+^{J_2 J_1} : \mathcal{K}_+^{J_1} \rightarrow \mathcal{K}_+^{J_2}, \quad T_-^{J_2 J_1} : -\mathcal{K}_-^{J_1} \rightarrow -\mathcal{K}_-^{J_2}.$$

It follows immediately that their direct sum map $T_+^{J_2 J_1} \oplus T_-^{J_2 J_1} : |\mathcal{K}|^{J_1} \rightarrow |\mathcal{K}|^{J_2}$ is a bijective linear continuous map with continuous inverse and hence as Banach spaces $|\mathcal{K}|^{J_1}$ and $|\mathcal{K}|^{J_2}$ have equivalent norms. \square

Proposition 3.7. *Let $\mathcal{K}_A = \mathcal{K}_+^J \oplus \mathcal{K}_-^J$ be the fundamental decomposition of the right (left) unital Kreĭn C^* -module \mathcal{K}_A associated to the fundamental symmetry J and with associated Hilbert C^* -module $|\mathcal{K}|_A^J$. The operator $T : \mathcal{K}_A \rightarrow \mathcal{K}_A$ is an adjointable operator in \mathcal{K}_A if and only if it is adjointable in $|\mathcal{K}|_A^J$. The adjoint T^* of T in the Kreĭn C^* -module \mathcal{K}_A and the adjoint $T^{\dagger J}$ of T in the Hilbert C^* -module $|\mathcal{K}|_A^J$ are related by the following formulas:*

$$T^{\dagger J} = J \circ T^* \circ J, \quad T^* = J \circ T^{\dagger J} \circ J.$$

The family $\mathcal{B}(\mathcal{K}_A)$ of adjointable operators in the unital right (left) Kreĭn C^ -module \mathcal{K}_A coincides as a set with the C^* -algebra $\mathcal{B}(|\mathcal{K}|_A^J)$ of adjointable operators in the unital right (left) Hilbert C^* -module $|\mathcal{K}|_A^J$.*

Proof. If T is adjointable in the Kreĭn C^* -module \mathcal{K}_A with adjoint T^* , we have, for all $x, y \in \mathcal{K}_A$, $\langle x | T(y) \rangle_{\mathcal{K}} = \langle T^*(x) | y \rangle_{\mathcal{K}}$ and so $\langle J(x) | T(y) \rangle_{|\mathcal{K}|^J} = \langle J(T^*(x)) | y \rangle_{|\mathcal{K}|^J}$ and taking $x := J(z)$ we get $\langle z | T(y) \rangle_{|\mathcal{K}|^J} = \langle J \circ T^* \circ J(z) | y \rangle_{|\mathcal{K}|^J}$ that gives the adjointability of T in $|\mathcal{K}|_A^J$ with adjoint $T^{\dagger J} = J \circ T^* \circ J$. Following the same passages in the reverse order establishes the equivalence of the notions of adjointability in \mathcal{K}_A and in $|\mathcal{K}|_A^J$. \square

Theorem 3.8. *The algebra $\mathcal{B}(\mathcal{K}_A)$ of adjointable endomorphisms of a unital right (left) Kreĭn C^* -module \mathcal{K}_A is a Kreĭn- C^* -algebra.*

Proof. By the previous proposition, we see that $\mathcal{B}(\mathcal{K}_A) = \mathcal{B}(|\mathcal{K}|_A^J)$ as sets and also as unital associative algebras, since the operations of addition and scalar multiplication are the same. Taking on $\mathcal{B}(\mathcal{K}_A)$ the operator norm $\|\cdot\|_J$ defined in the C^* -algebra $\mathcal{B}(|\mathcal{K}|_A^J)$ of the unital Hilbert C^* -module $|\mathcal{K}|_A^J$, we see that $\mathcal{B}(\mathcal{K}_A)$ is a Banach space. The involution on the algebra $\mathcal{B}(\mathcal{K}_A)$ is obtained via the Kreĭn C^* -module adjoint $T \mapsto T^*$. In order to complete the proof that $\mathcal{B}(\mathcal{K}_A)$ is a Kreĭn C^* -algebra, we need to provide an involutive automorphism $\alpha : \mathcal{B}(\mathcal{K}_A) \rightarrow \mathcal{B}(\mathcal{K}_A)$ such that $\|\alpha(T^*) \circ T\|_J = \|T\|_J^2$ for all $T \in \mathcal{B}(\mathcal{K}_A)$. For this purpose, for all $T \in \mathcal{B}(\mathcal{K}_A)$, we define $\alpha(T) := J \circ T \circ J$ and, using the C^* -property of $\mathcal{B}(|\mathcal{K}|_A^J)$, verify that $\|\alpha(T^*) \circ T\|_J = \|(J \circ T \circ J) \circ T\|_J = \|T^{\dagger J} \circ T\|_J = \|T\|_J^2$. \square

Proposition 3.9. *Let \mathcal{K}_A be a unital right (left) Kreĭn C^* -module. Any two fundamental symmetries $J_1, J_2 \in \mathcal{B}(\mathcal{K}_A)$ of \mathcal{K} are unitarily equivalent.*

Proof. With the notation used in theorem 3.5, consider the adjointable operator $U := T_+^{J_2 J_1} \oplus T_-^{J_2 J_1}$ and note that $U \circ J_1 = J_2 \circ U$ with $U^* = U^{-1}$. \square

Example 3.10. Every Kreĭn-space is a Kreĭn- C^* -module over the C^* -algebra \mathbb{C} , the complexification $\mathbb{M}_{\mathbb{C}} := \mathbb{M} \otimes_{\mathbb{R}} \mathbb{C}$ of Minkowski space in special relativity being probably the most important example. \lrcorner

Example 3.11. Let M be a semi-Riemannian manifold that for now (although this makes the situation less interesting for applications to physics) we suppose to be compact. If the manifold is also supposed to be space-orientable and time-orientable (for a specific example, consider \mathbb{T}^2 with the indefinite metric coming from the product of a copy of \mathbb{T} with positive metric and another copy with negative metric), the module $\Gamma(T(M))$ of continuous sections of its tangent bundle $T(M)$ is a unital Kreĭn C^* -module over the unital C^* -algebra $C(M)$ of continuous functions.

Note that, when a semi-Riemannian manifold is not space-orientable and time-orientable, although each fiber of the tangent bundle $T(M)$ is still a Kreĭn space, the module $\Gamma(T(M))$ of continuous vector fields, fails to be a Kreĭn C^* -module over the C^* -algebra $C(M)$, because in this case $\Gamma(T(M))$ does not admit a global splitting as a direct sum of a Hilbert and an anti-Hilbert C^* -module (equivalently the tangent bundle $T(M)$ is not a Whitney direct sum of a positive definite and a negative definite sub-bundle).

As a consequence, this very interesting kind of complete semi-definite Hilbert C^* -modules do not admit a globally defined fundamental symmetry! \lrcorner

4 Kreĭn C*-modules over Kreĭn C*-algebras

Definition 4.1. A *left Kreĭn C*-module* over a Kreĭn C*-algebra \mathcal{A} is a complete topological vector space that is also a left (unital) module \mathcal{K} over \mathcal{A} with the the following properties:

- \mathcal{K} is equipped with an \mathcal{A} -valued inner product ${}_{\mathcal{A}}(\cdot | \cdot) : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{A}$ such that

$$\begin{aligned} {}_{\mathcal{A}}(x + y | z) &= {}_{\mathcal{A}}(x | z) + {}_{\mathcal{A}}(y | z), \quad \forall x, y, z \in \mathcal{K}, \\ {}_{\mathcal{A}}(ax | y) &= a \cdot {}_{\mathcal{A}}(x | y), \quad \forall x, y \in \mathcal{K}, \quad \forall a \in \mathcal{A}, \\ {}_{\mathcal{A}}(x | y)^* &= {}_{\mathcal{A}}(y | x), \quad \forall x, y \in \mathcal{K}, \\ \forall y \in \mathcal{K}, \quad {}_{\mathcal{A}}(x | y) &= 0 \Rightarrow x = 0, \end{aligned}$$

- there exists a **fundamental symmetry** $J_{\mathcal{A}} : \mathcal{K} \rightarrow \mathcal{K}$ such that $J \circ J = \text{Id}_{\mathcal{K}}$,

$$\begin{aligned} J_{\mathcal{A}}(x + y) &= J_{\mathcal{A}}(x) + J_{\mathcal{A}}(y), \quad \forall x, y \in \mathcal{K}, \\ J_{\mathcal{A}}(a \cdot x) &= \alpha(a) \cdot J_{\mathcal{A}}(x), \quad \forall x \in \mathcal{K}, \quad \forall a \in \mathcal{A}, \\ \alpha({}_{\mathcal{A}}(x | y)) &= {}_{\mathcal{A}}(J_{\mathcal{A}}(x) | J_{\mathcal{A}}(y)), \quad \forall x, y \in \mathcal{K}, \end{aligned}$$

- for the given choice of fundamental symmetries α on \mathcal{A} and $J_{\mathcal{A}}$ on \mathcal{K} , we have that the map $(x, y) \mapsto {}_{\mathcal{A}}(x | J_{\mathcal{A}}(y))$ gives to \mathcal{K} the structure of a Hilbert C*-module over the C*-algebra \mathcal{A}^{α} whose norm induces the original topology of \mathcal{K} .

In a perfectly similar way, there is a definition of **right Kreĭn C*-module** \mathcal{K} over a Kreĭn C*-algebra \mathcal{B} , where the \mathcal{B} -valued inner product $(\cdot | \cdot)_{\mathcal{B}} : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{B}$ satisfies $(x | yb)_{\mathcal{B}} = (x | y)_{\mathcal{B}} \cdot b$, for all $b \in \mathcal{B}$ and all $x, y \in \mathcal{K}$; the fundamental symmetry $J_{\mathcal{B}}$ satisfies $J_{\mathcal{B}}(x \cdot b) = J_{\mathcal{B}}(x) \cdot \beta(b)$, for all $x \in \mathcal{K}$ and $b \in \mathcal{B}$; and where all the other properties remain essentially unchanged.

The following definition of bimodule, whenever we further impose the additional fullness requirements $\mathcal{A} = {}_{\mathcal{A}}(\mathcal{K} | \mathcal{K}) := \text{span}\{{}_{\mathcal{A}}(x | y) \mid x, y \in \mathcal{K}\}$ and $\mathcal{B} = (\mathcal{K} | \mathcal{K})_{\mathcal{B}} := \text{span}\{(x | y)_{\mathcal{B}} \mid x, y \in \mathcal{K}\}$, extends to the Kreĭn C*-algebras context the usual notion of imprimitivity Hilbert C*-bimodule that provides the well-known (strong) Morita equivalence of C*-algebras (see also remark 5.7).

Definition 4.2. A **Kreĭn C*-bimodule** ${}_{\mathcal{A}}\mathcal{K}_{\mathcal{B}}$ is a left Kreĭn C*-module over \mathcal{A} that is the same time a right Kreĭn C*-module over \mathcal{B} with additional properties: $(a \cdot x) \cdot b = a \cdot (x \cdot b)$, $J_{\mathcal{A}} = J_{\mathcal{B}}$ and with right and left inner products related via: ${}_{\mathcal{A}}(x | y)x = x(y | x)_{\mathcal{B}}$, for all $x, y \in \mathcal{K}$.

The compatibility condition requested above on the inner products assures that the induced left and right norms on the Hilbert C*-module \mathcal{K} coincide.

Remark 4.3. In our definitions the two auxiliary \mathcal{A}^{α} -valued Hilbert C*-module inner products

$$(x | y)_{\mathcal{A}^{\alpha}}^{J_{\mathcal{A}}} := (x | J_{\mathcal{A}}(y))_{\mathcal{A}}, \quad {}^{J_{\mathcal{A}}}(x | y)_{\mathcal{A}^{\alpha}} := (J_{\mathcal{A}}(x) | y)_{\mathcal{A}},$$

can be used in place of each other since they are related by the isomorphism α of the C*-algebra \mathcal{A}^{α} : $\alpha((J_{\mathcal{A}}(x) | y)_{\mathcal{A}}) = (x | J_{\mathcal{A}}(y))_{\mathcal{A}}$. \lrcorner

Remark 4.4. Note that $\mathcal{K}_{\mathcal{B}}$ becomes naturally a Kreĭn C*-module over the C*-algebra \mathcal{B}_{+} when equipped with the even part of the original inner product i.e. taking $\frac{1}{2}(x | y)_{\mathcal{B}} + \frac{1}{2}(J(x) | J(y))_{\mathcal{B}}$ as inner product on \mathcal{K} ; however with this new inner product the submodules \mathcal{K}_{+} and \mathcal{K}_{-} are orthogonal (as in our original definition of Kreĭn C*-module over a C*-algebra) contrary to the general situation here, where the obstruction to the orthogonality is measured by the odd part of the original inner product $\frac{1}{2}(J(x) | y)_{\mathcal{B}} + \frac{1}{2}(x | J(y))_{\mathcal{B}}$ that is always a non-degenerate anti-Hermitian product on \mathcal{K}

with values in \mathcal{B}_- . Furthermore, although in some situations (for example in finite dimensional cases) the product topology induced by the decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ (that is actually the topology induced by the even part of the inner product, since the two inner products coincide, modulo sign, on their restriction to \mathcal{K}_+ and \mathcal{K}_- respectively) is the same as the original topology of \mathcal{K} , we still suspect that in general this might fail.

A significant difference from the case of Kreĭn C^* -modules over C^* -algebras is that the fundamental symmetry $J_{\mathcal{B}}$ is in general not self-adjoint (and in general not even adjointable, as can be seen in the case of example 4.7 below, where adjointability happens only in the case of α equal to the identity i.e. when \mathcal{A} is already a C^* -algebra) with respect to the original Kreĭn inner product as suggested by the general lack of orthogonality between the even and odd submodules \mathcal{K}_+ and \mathcal{K}_- . \lrcorner

Example 4.5. Every Kreĭn C^* -module \mathcal{K} over a C^* -algebra \mathcal{A} , as defined in the previous section, is a special case of our new definition as results by taking α to be the identity isomorphism of \mathcal{A} . Actually, whenever the Kreĭn C^* -algebra \mathcal{A} is a C^* -algebra and α is trivial, we reduce to the definition of the last section: the submodules \mathcal{K}_+ and \mathcal{K}_- are orthogonal and the fundamental symmetry $J_{\mathcal{A}}$ is Hermitian. The new definition of Kreĭn module over a Kreĭn C^* -algebra even allows for a possible choice of a nontrivial α also in the case of a C^* -algebra \mathcal{A} and in this situation we obtain a Kreĭn C^* -module over a C^* -algebra where the two submodules \mathcal{K}_+ and \mathcal{K}_- fail to be orthogonal and $J_{\mathcal{A}}$ is not Hermitian. \lrcorner

Example 4.6. Let $\mathcal{K}_{\mathcal{A}}$ be a right Kreĭn C^* -module over the C^* -algebra \mathcal{A} and let $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ be the Kreĭn C^* -algebra of adjointable operators on \mathcal{K} , then ${}_{\mathcal{B}(\mathcal{K})}\mathcal{K}$ is a left Kreĭn C^* -module over the Kreĭn C^* -algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ with left inner product defined by ${}_{\mathcal{B}(\mathcal{K})}(x | y) := \Theta_{x,y}$, where $\Theta_{x,y}(z) := x \cdot (y | z)_{\mathcal{A}}$, for all $x, y, z \in \mathcal{K}$.

The bimodule ${}_{\mathcal{B}(\mathcal{K})}\mathcal{K}_{\mathcal{A}}$ is actually a Kreĭn C^* -bimodule such that ${}_{\mathcal{B}(\mathcal{K})}(x | y)z = x(y | z)_{\mathcal{A}}$ for all $x, y, z \in \mathcal{K}$. \lrcorner

Example 4.7. Let \mathcal{A} be a Kreĭn C^* -algebra. Then ${}_{\mathcal{A}}\mathcal{A}$ and $\mathcal{A}_{\mathcal{A}}$ are both Kreĭn C^* -modules with inner products given by $(x | y)_{\mathcal{A}} := x^*y$ and ${}_{\mathcal{A}}(x | y) := xy^*$, furthermore ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ is a Kreĭn C^* -bimodule. \lrcorner

Example 4.8. Let K_1 and K_2 be two Kreĭn spaces.

The space ${}_{\mathcal{B}(K_2)}\mathcal{B}(K_1, K_2)_{\mathcal{B}(K_1)}$ of linear continuous maps between them is a Kreĭn C^* -bimodule with the left/right actions given by the usual compositions of linear operators and inner products given respectively by $(T | S)_{\mathcal{B}(K_1)} := T^* \circ S$ and ${}_{\mathcal{B}(K_2)}(T | S) := T \circ S^*$. \lrcorner

Example 4.9. Following the definitions provided in [BR], let $A, B \in \text{Ob}_{\mathcal{A}}$ be two objects in a Kreĭn C^* -category \mathcal{A} . Then $\mathcal{A}_{AB} := \text{Hom}_{\mathcal{A}}(B, A)$ is a Kreĭn C^* -bimodule over the Kreĭn C^* -algebras \mathcal{A}_{AA} on the left and \mathcal{A}_{BB} on the right. \lrcorner

Example 4.10. Let \mathbb{M} be Minkowski space (or more generally any real vector space equipped with semi-definite inner product); let $\Lambda^{\mathbb{C}}(\mathbb{M})$ denote the space of complex-valued antisymmetric forms on \mathbb{M} (the complexified Grassmann algebra of \mathbb{M}) and let $\text{Cl}(\mathbb{M})$ denote the complexified Clifford algebra of \mathbb{M} .

Note that for every fundamental decomposition of $\mathbb{M} = \mathbb{M}_+ \oplus \mathbb{M}_-$, we have for the Grassmann algebras the decomposition $\Lambda^{\mathbb{C}}(\mathbb{M}) := \Lambda^{\mathbb{C}}(\mathbb{M}_+) \hat{\otimes} \Lambda^{\mathbb{C}}(\mathbb{M}_-)$ and similarly, for the Clifford algebras, $\text{Cl}(\mathbb{M}) = \text{Cl}(\mathbb{M}_+) \hat{\otimes} \text{Cl}(\mathbb{M}_-)$, where (if we work in the category of associative algebras) $\hat{\otimes}$ denotes the \mathbb{Z}_2 -graded tensor product.

Note that the (underlying complex vector space of the) Grassmann algebra $\Lambda^{\mathbb{C}}(\mathbb{M})$ is naturally a Kreĭn space with the semi-definite inner product induced by universal factorization property via $(\omega_1 \wedge \cdots \wedge \omega_n | \phi_1 \wedge \cdots \wedge \phi_n) := \det[(\omega_i | \phi_j)]$ on each of the summands in $\bigoplus_{q=0}^{\dim \mathbb{M}} \Lambda_q^{\mathbb{C}}(\mathbb{M}) = \Lambda^{\mathbb{C}}(\mathbb{M})$, where $(\omega_i | \phi_j) \in \mathbb{C}$ denotes the Minkowski inner product on complexified covectors $\omega_i, \phi_j \in \Lambda_1^{\mathbb{C}}(\mathbb{M})$.

The Clifford algebra $\mathbb{Cl}(\mathbb{M})$ has a natural structure of Kreĭn C^* -algebra as a sub-algebra of the Kreĭn C^* -algebra $\mathcal{B}(\Lambda^{\mathbb{C}}(\mathbb{M}))$: every fundamental symmetry $J_{\mathbb{M}}$ of Minkowski space lifts to an involutive automorphism of $\mathbb{Cl}(\mathbb{M})$ (by the defining universal property of the Clifford algebra) that, under the linear isomorphism $\mathbb{Cl}(\mathbb{M}) \simeq \Lambda^{\mathbb{C}}(\mathbb{M})$, coincides with the (second quantized) fundamental symmetry $J_{\Lambda^{\mathbb{C}}(\mathbb{M})} := \bigoplus_{q=0}^{\infty} J_{\mathbb{M}}^{\wedge q}$ of the Kreĭn space $\Lambda^{\mathbb{C}}(\mathbb{M})$. It follows that the Kreĭn space $\Lambda^{\mathbb{C}}(\mathbb{M})$ is a left Kreĭn module over the Kreĭn C^* -algebra $\mathbb{Cl}(\mathbb{M})$.

The (underlying vector space of the) Grassmann algebra $\Lambda^{\mathbb{C}}(\mathbb{M})$ also becomes a Kreĭn C^* -bimodule over $\mathbb{Cl}(\mathbb{M})$ via Clifford left and right actions and with the inner products induced via the linear isomorphism $\Lambda^{\mathbb{C}}(\mathbb{M}) \simeq \mathbb{Cl}(\mathbb{M})$ by the standard Kreĭn C^* -bimodule structure of the Kreĭn C^* -algebra $\mathbb{Cl}(\mathbb{M})$ over itself.

As described in more detail in H.Baum [Ba] (see also A.Strohmaier [Str, section 5.1] and K.Van Den Dungen-M.Paschke-A.Rennie [DPR, section 3.3.1]) the module $S(\mathbb{M})$ of (Dirac) spinors is a Kreĭn space, whose fundamental symmetries are proportional to the product of the operators of Clifford multiplication by all the vectors in an orthonormal basis for the timelike summand of a fundamental decomposition $\mathbb{M} = \mathbb{M}_+ \oplus \mathbb{M}_-$.⁴ The space $S(\mathbb{M})$ (for \mathbb{M} even-dimensional) becomes a left Kreĭn C^* -module over the Kreĭn C^* -algebra $\mathbb{Cl}(\mathbb{M})$ with the inner product induced by the linear isomorphisms $S(\mathbb{M}) \otimes S(\mathbb{M})^* \simeq \Lambda^{\mathbb{C}}(\mathbb{M}) \simeq \mathbb{Cl}(\mathbb{M})$ and ${}_{\mathbb{Cl}(\mathbb{M})}S(\mathbb{M})_{\mathbb{C}}$ is a Kreĭn C^* -bimodule that, with the terminology introduced in remark 5.7, is an example of Morita-Kreĭn equivalence C^* -bimodule. \lrcorner

Example 4.11. Let M be a (compact) semi-Riemannian space-orientable and time-orientable manifold. As already described in example 3.11, the module $\Gamma(T(M))$ of its continuous vector fields is a (unital) Kreĭn C^* -bimodule over the (unital) C^* -algebra $C(M)$. The algebra $\Gamma(\mathbb{Cl}(M))$ of continuous section of the complexified Clifford bundle $\mathbb{Cl}(T(M))$ of M is a (unital) Kreĭn C^* -algebra and the module $\Gamma(\Lambda^{\mathbb{C}}(M))$ of continuous sections of the complexified Grassmann bundle $\Lambda^{\mathbb{C}}(M)$ of M is a (unital) Kreĭn C^* -bimodule over the Kreĭn C^* -algebra $\Gamma(\mathbb{Cl}(M))$. The case of spinorial manifolds is described in example 5.9. \lrcorner

Remark 4.12. Note that, in the previous example, if the manifold M is not time-orientable and space-orientable, the algebra $\Gamma(\mathbb{Cl}(M))$ (although being a nice involutive complete topological algebra) does not admit a globally defined fundamental symmetry and so does not fit into the current definition of Kreĭn C^* -algebra!

This clearly indicates that the environment of Kreĭn C^* -algebras and Kreĭn C^* -modules that we have developed here is insufficient to deal with a general axiomatization of “complete semi-definite C^* -algebras and C^* -modules over them”. \lrcorner

We pass now to briefly examine the main properties of the algebras of adjointable operators on Kreĭn C^* -modules over Kreĭn C^* -algebras.

Definition 4.13. Let $\mathcal{K}_{\mathcal{A}}$ be a Kreĭn C^* -module over the Kreĭn C^* -algebra \mathcal{A} . A map $T : \mathcal{K} \rightarrow \mathcal{K}$ is said to be **adjointable** if there exists another map $T^* : \mathcal{K} \rightarrow \mathcal{K}$ such that $(T(x) \mid y)_{\mathcal{A}} = (x \mid T^*(y))_{\mathcal{A}}$, for all $x, y \in \mathcal{K}$. The family of such adjointable maps is denoted by $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$.

Remark 4.14. As usual, the adjointable maps are already \mathcal{A} -linear and continuous and the adjoint T^* is unique. The set $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ is a vector space and an associative unital algebra under composition, furthermore the map $*$: $T \mapsto T^*$ is involutive, antimultiplicative and conjugate \mathbb{C} -linear so that $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ is a complex associative unital $*$ -algebra. \lrcorner

Proposition 4.15. A map $T : \mathcal{K}_{\mathcal{A}} \rightarrow \mathcal{K}_{\mathcal{A}}$ is adjointable with respect to the inner product $(\cdot \mid \cdot)_{\mathcal{A}}$ if and only if the map T is adjointable for the Hilbert C^* -module $\mathcal{K}_{\mathcal{A}^{\alpha}}$ with the auxiliary inner product

⁴On the usual Minkowski space \mathbb{M}^4 , the Kreĭn space $S(\mathbb{M}^4)$ has signature $(2, 2)$ and the fundamental symmetries are just the Dirac γ^0 operators.

$(\cdot | \cdot)_{\mathcal{K}_{\mathcal{A}^\alpha}}^{J_{\mathcal{A}}}$. As a consequence, the associative unital algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ coincides with the associative unital algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}^\alpha})$. The relation between the adjoint T^* of T in $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ and the adjoint T^{\dagger_α} of T in the C^* -algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}^\alpha})$ is given by:

$$T^{\dagger_\alpha} = J_{\mathcal{A}} \circ T^* \circ J_{\mathcal{A}}, \quad T^* = J_{\mathcal{A}} \circ T^{\dagger_\alpha} \circ J_{\mathcal{A}}$$

Proof. Suppose that $(T(x) | y)_{\mathcal{A}} = (y | T^*(y))_{\mathcal{A}}$. The following calculation

$$\begin{aligned} (T(x) | y)_{\mathcal{K}_{\mathcal{A}^\alpha}}^{J_{\mathcal{A}}} &= (T(x) | J_{\mathcal{A}}(y))_{\mathcal{A}} = (x | T^* J_{\mathcal{A}}(y))_{\mathcal{A}} \\ &= (x | J_{\mathcal{A}} J_{\mathcal{A}} T^* J_{\mathcal{A}}(y))_{\mathcal{A}} = (x | J_{\mathcal{A}} T^* J_{\mathcal{A}}(y))_{\mathcal{K}_{\mathcal{A}^\alpha}}^{J_{\mathcal{A}}}, \end{aligned}$$

assures that the adjointability in $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ implies the adjointability in $\mathcal{B}(\mathcal{K}_{\mathcal{A}^\alpha})$ and the first formula relating the adjoints.

Suppose now that $(T(x) | y)_{\mathcal{K}_{\mathcal{A}^\alpha}}^{J_{\mathcal{A}}} = (x | T^{\dagger_\alpha}(y))_{\mathcal{K}_{\mathcal{A}^\alpha}}^{J_{\mathcal{A}}}$ i.e. $(T(x) | J_{\mathcal{A}}(y))_{\mathcal{A}} = (x | J_{\mathcal{A}} T^{\dagger_\alpha}(y))_{\mathcal{A}}$ and choosing $y = J_{\mathcal{A}}(y')$ for an arbitrary $y' \in \mathcal{K}$, we obtain $(T(x) | (y'))_{\mathcal{A}} = (x | J_{\mathcal{A}} T^{\dagger_\alpha} J_{\mathcal{A}}(y'))_{\mathcal{A}}$ that assures the reverse and the second adjointability formula. \square

Although we know that in general $J_{\mathcal{A}}$ is not an adjointable operator, we still have the following fundamental symmetry of $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$:

Proposition 4.16. *If T is adjointable in $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$, also the new operator $J_{\mathcal{A}} \circ T \circ J_{\mathcal{A}}$ is adjointable in $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ and the map*

$$\alpha_{J_{\mathcal{A}}} : T \mapsto J_{\mathcal{A}} \circ T \circ J_{\mathcal{A}}$$

is a $$ -isomorphism of the involutive algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ of adjointable operators.*

Proof. Since, for all $T \in \mathcal{B}(\mathcal{K}_{\mathcal{A}})$, $(T^*)^* = T$, we obtain $J_{\mathcal{A}}(J_{\mathcal{A}} T^{\dagger_\alpha} J_{\mathcal{A}})^{\dagger_\alpha} J_{\mathcal{A}} = T$ or equivalently $(J_{\mathcal{A}} T^{\dagger_\alpha} J_{\mathcal{A}})^{\dagger_\alpha} = J_{\mathcal{A}} T J_{\mathcal{A}}$. Since $(S^{\dagger_\alpha})^{\dagger_\alpha} = S$, we get $(J_{\mathcal{A}} T^{\dagger_\alpha} J_{\mathcal{A}}) = (J_{\mathcal{A}} T J_{\mathcal{A}})^{\dagger_\alpha}$ i.e. $\alpha_{J_{\mathcal{A}}}$ is a \dagger_α -isomorphism of the C^* -algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}^\alpha})$.

Similarly from $(S^{\dagger_\alpha})^{\dagger_\alpha} = S$, we get $J_{\mathcal{A}}(J_{\mathcal{A}} S^* J_{\mathcal{A}})^* J_{\mathcal{A}} = S$ or equivalently $(J_{\mathcal{A}} S^* J_{\mathcal{A}})^* = J_{\mathcal{A}} S J_{\mathcal{A}}$ and hence $J_{\mathcal{A}} S^* J_{\mathcal{A}} = (J_{\mathcal{A}} S J_{\mathcal{A}})^*$ i.e. $\alpha_{J_{\mathcal{A}}}$ is a $*$ -isomorphism of the involutive algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$. \square

Theorem 4.17. *The algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$ of adjointable operators of a Kreĭn C^* -module over a Kreĭn C^* -algebra \mathcal{A} is a Kreĭn C^* -algebra.*

Proof. The $*$ -isomorphism $\alpha_{J_{\mathcal{A}}}$ of the involutive algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}})$, defined in the previous proposition, satisfies the C^* -property $\|\alpha_{J_{\mathcal{A}}}(T^*)T\|_\alpha = \|T\|_\alpha^2$ with respect to the norm of the C^* -algebra $\mathcal{B}(\mathcal{K}_{\mathcal{A}^\alpha})$. \square

5 Categories of Kreĭn C^* -modules

The following proposition, whose proof is self-evident, provides the most elementary category of morphisms of Kreĭn C^* -algebras that naturally contains, as a full subcategory, the category of unital $*$ -homomorphisms of unital C^* -algebras.

Proposition 5.1. *There is a category \mathcal{A} whose objects are unital Kreĭn C^* -algebras $\mathcal{A}, \mathcal{B}, \dots$; whose arrows $\phi : \mathcal{A} \rightarrow \mathcal{B}$ are unital Kreĭn $*$ -homomorphisms i.e. unital $*$ -homomorphisms of involutive unital algebras $\phi : \mathcal{A} \rightarrow \mathcal{B}$ such that there exist at least a fundamental symmetry of α of \mathcal{A} and β of \mathcal{B} such that $\phi \circ \alpha = \beta \circ \phi$; and composition is the usual composition of functions.*

We now define the Kreĭn C^* -analogue of the well-known notion of C^* -correspondence.

Definition 5.2. A *left Kreĭn C*-correspondence* from the unital Kreĭn C*-algebra \mathcal{B} to unital the Kreĭn C*-algebra \mathcal{A} is unital left Kreĭn C*-module ${}_A\mathcal{M}$ over the Kreĭn C*-algebra \mathcal{A} equipped with a morphism of unital Kreĭn C*-algebras from \mathcal{B} to the unital Kreĭn C*-algebra $\mathcal{B}({}_A\mathcal{M})$ of adjointable operators on the Kreĭn module ${}_A\mathcal{M}$ such that $x \cdot \beta(b) = J_A(J_A(x) \cdot b)$.

A *right Kreĭn C*-correspondence* from \mathcal{B} to \mathcal{A} is similarly defined as a unital right Kreĭn C*-module \mathcal{N}_B over \mathcal{B} equipped with a morphism of unital Kreĭn C*-algebras from \mathcal{A} to $\mathcal{B}(\mathcal{N}_B)$ such that $\alpha(a) \cdot x = J_B(a \cdot (J_B x))$.

A *morphism of Kreĭn C*-correspondences* is a map $\Phi : {}_A\mathcal{M}_B \rightarrow {}_A\mathcal{N}_B$ between right (respectively left) Kreĭn C*-correspondences such that

$$\Phi(a \cdot x \cdot b) = a \cdot \Phi(x) \cdot b, \quad \forall a \in \mathcal{A}, b \in \mathcal{B}, x \in \mathcal{M}.$$

Remark 5.3. The previous definition entails that a left Kreĭn C*-correspondence is actually a unital bimodule ${}_A\mathcal{M}_B$ over the the unital Kreĭn C*-algebras \mathcal{A} and \mathcal{B} (with \mathcal{A} -valued inner product) such that there exists at least one fundamental symmetry J of ${}_A\mathcal{M}$ and fundamental symmetries α of \mathcal{A} , β of \mathcal{B} that satisfy the compatibility condition $J(axb) = \alpha(a)J(x)\beta(b)$, for all $x \in \mathcal{M}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Clearly we have categories of morphisms of right (respectively) Kreĭn C*-correspondences under the usual composition of morphisms. \lrcorner

The following definition and theorems incorporate (and generalize to the case of Kreĭn C*-modules over Kreĭn C*-algebras) the notion of tensor product of Kreĭn spaces and Kreĭn C*-modules over C*-algebras developed in R.Tanadkithirun's senior undergraduate project [T].

Definition 5.4. The *internal tensor product of two right Kreĭn C*-correspondences* ${}_A\mathcal{M}_B$ and ${}_B\mathcal{N}_C$ is defined as a left \mathcal{A} -linear right \mathcal{C} -linear and \mathcal{B} -balanced map $\otimes : {}_A\mathcal{M}_B \times {}_B\mathcal{N}_C \rightarrow {}_A\mathcal{T}_C$ with values into a right Kreĭn C*-correspondence ${}_A\mathcal{T}_B$ from \mathcal{A} to \mathcal{B} such that the following universal factorization property is satisfied:

for every left \mathcal{A} -linear right \mathcal{C} -linear and \mathcal{B} -balanced function $\phi : \mathcal{M} \times \mathcal{N} \rightarrow \mathcal{Q}$ with values into a Kreĭn C*-correspondence ${}_A\mathcal{Q}_C$ from \mathcal{A} to \mathcal{C} , there exists a unique morphism $\Phi : \mathcal{T} \rightarrow \mathcal{Q}$ of Kreĭn C*-correspondences such that $\Phi \circ \otimes = \phi$.

Theorem 5.5. Tensor products of right Kreĭn C*-correspondences exist and are unique up to isomorphism in the category of morphisms of Kreĭn C*-correspondences.

Proof. The unicity up to isomorphism is a standard consequence of a definition via universal factorization properties. For the proof of existence, consider the fundamental decompositions $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$ and $\mathcal{N} = \mathcal{N}_+ \oplus \mathcal{N}_-$ induced by a pair of fundamental symmetries J_M and J_N that are compatible with three fundamental symmetries α of \mathcal{A} , β of \mathcal{B} and γ of \mathcal{C} .

Using the algebraic tensor product of bimodules and the canonical isomorphism of bimodules

$$(\mathcal{M}_+ \oplus \mathcal{M}_-) \otimes_B (\mathcal{N}_+ \oplus \mathcal{N}_-) \simeq [(\mathcal{M}_+ \otimes_B \mathcal{N}_+) \oplus (\mathcal{M}_- \otimes_B \mathcal{N}_-)] \oplus [(\mathcal{M}_+ \otimes_B \mathcal{N}_-) \oplus (\mathcal{M}_- \otimes_B \mathcal{N}_+)], \quad (5.1)$$

we have that $J_M \otimes_B J_N$ is a fundamental symmetry of $\mathcal{M} \otimes_B \mathcal{N}$ that induces the previous decomposition and is compatible with the left action of \mathcal{A} and the right action of \mathcal{C} .

By universal factorization property (two times), we can define on $\mathcal{M} \otimes_B \mathcal{N}$ a unique \mathcal{C} -valued inner product such that, for all $x_1, x_2 \in \mathcal{M}$ and $y_1, y_2 \in \mathcal{N}$,

$$(x_1 \otimes_B y_1 \mid x_2 \otimes_B y_2)_{\mathcal{C}}^{\mathcal{M} \otimes_B \mathcal{N}} := (y_1 \mid (x_1 \mid x_2)_{\mathcal{B}}^{\mathcal{M}} \cdot y_2)_{\mathcal{C}}^{\mathcal{N}}.$$

The following property holds for this inner product

$$\gamma((x_1 \otimes_B y_1 \mid x_2 \otimes_B y_2)_{\mathcal{C}}) = ((J_M \otimes_B J_N)(x_1 \otimes_B y_1) \mid (J_M \otimes_B J_N)(x_2 \otimes_B y_2))_{\mathcal{C}}$$

and the algebra \mathcal{A} acts by adjointable operators on the left.

The inner product so defined on $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$ induces on each one of the direct summands of the even part of the decomposition in formula (5.1) the structure of Hilbert C^* -module and, for the summands of the odd part, the structure of anti-Hilbert C^* -module over the same C^* -algebra $(\mathcal{C}, \dagger_\gamma)$. \square

Theorem 5.6. *There is a weak category \mathcal{M}_\bullet whose objects are unital Kreĭn C^* -algebras $\mathcal{A}, \mathcal{B}, \dots$; whose arrows are right Kreĭn C^* -correspondences; and whose composition is obtained by internal tensor product of Kreĭn C^* -correspondences. In a totally similar way, we have a weak category $\bullet\mathcal{M}$ of left Kreĭn C^* -correspondences under internal tensor product.*

Proof. The associativity of composition modulo isomorphism is assured using the universal factorization property. The (weak) identities are given the Kreĭn C^* -algebras \mathcal{A} considered as right Kreĭn C^* -correspondences over themselves when equipped with their standard right inner product $(a_1 \mid a_2)_{\mathcal{A}} := a_1^* a_2$, $a_1, a_2 \in \mathcal{A}$. \square

Remark 5.7. The previous categories \mathcal{M}_\bullet and $\bullet\mathcal{M}$ are actually 2-categories considering as 2-arrows the morphisms of Kreĭn C^* -correspondences with their usual functional composition as composition over 1-arrows and their internal tensor product as composition over objects.

This pair of weak 2-categories is the “Kreĭn counterpart” to the usual 2-categories of right and left C^* -correspondences and their (common) subcategory of 1-isomorphisms, that consists of Kreĭn C^* -bimodules ${}_A\mathcal{M}_B$ that are full and satisfy the imprimitivity condition ${}_A(x \mid y)z = x(y \mid z)_{\mathcal{B}}$, for all $x, y, z \in \mathcal{M}$, is the “Kreĭn counterpart” of the Morita-Rieffel weak category of imprimitivity C^* -bimodules that describe the (strong) Morita equivalence between C^* -algebras⁵ and hence we have a theoretical background capable of discussing the notion of “**Kreĭn-Morita equivalence**” at least in the context of Kawamura’s Kreĭn C^* -algebras. \lrcorner

Remark 5.8. The previous categories (exactly as their C^* -counterparts) are not equipped with involutions: the contragredient of a right correspondence is a left correspondence, but usually not another right correspondence.⁶ More interesting notions of “bivariant” Kreĭn C^* -bimodules will be developed elsewhere. \lrcorner

Example 5.9. Whenever the time-orientable space-orientable (compact) semi-Riemannian even-dimensional manifold M admits a spinorial structure, or more generally a spin^c structure, (see details in H.Baum [Ba]) the family $\Gamma(S(M))$ of continuous section of a given complex spinor bundle $S(M)$ becomes a Kreĭn-Morita equivalence Kreĭn C^* -bimodule between $C(M)$ (on the right) and the Kreĭn C^* -algebra $\Gamma(\text{Cl}(M))$ on the left.⁷ Its contragredient Kreĭn C^* -bimodule $\Gamma(S(M))^*$ is isomorphic to the Kreĭn C^* -bimodule of sections of the dual spinor bundle $S(M)^*$ and we have $\Gamma(S(M)) \otimes_{C(M)} \Gamma(S(M))^* \simeq \Gamma(\Lambda^{\mathbb{C}}(M))$ as tensor product of Kreĭn C^* -bimodules. \lrcorner

6 Outlook

The discussions of duality and of spectral theory, via suitable “Kreĭn bundles”, for some “commutative” subclasses of the Kreĭn C^* -modules here defined, will be dealt with in future works.⁸

⁵For additional details on the Morita-Rieffel categories and strong Morita equivalence see for example the review sections in [BCL] and the references therein.

⁶Recall that, given a bimodule ${}_A\mathcal{K}_B$, over involutive algebras \mathcal{A}, \mathcal{B} , its contragredient bimodule ${}_B\overline{\mathcal{K}}_A$ is the same Abelian group $\mathcal{K} := \mathcal{K}$ with left/right actions defined via $b \cdot \overline{x} \cdot a := \overline{a^* x b^*}$, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $x \in \mathcal{K}$. For a right Kreĭn C^* -correspondence \mathcal{K}_B with right inner product $(x \mid y)_{\mathcal{B}}$, for $x, y \in \mathcal{K}$, the contragredient ${}_B\overline{\mathcal{K}}$ is naturally a left Kreĭn C^* -correspondence via ${}_B(\overline{x} \mid \overline{y}) := (x \mid y)_{\mathcal{B}}$, for all $\overline{x}, \overline{y} \in \overline{\mathcal{K}}$. A similar statement holds for left correspondences.

⁷A similar statement holds in the odd-dimensional case if the Clifford algebra $\Gamma(\text{Cl}(M))$ is replaced by its even part $\Gamma(\text{Cl}^+(M))$, see [GVF, section 9.2].

⁸See anyway [BBL] for some elementary results in the case of commutative Kreĭn C^* -algebras.

The notion of Krein C^* -module over a Krein C^* -algebra that we presented here, although interesting as a first step to explore some of the issues in semi-definite situations, is still too elementary to be fully useful for general applications to non-commutative spectral geometry, at least whenever the semi-Riemannian geometry involved presents topological obstructions to orientability, either in spacelike or in timelike sense (or both). Since global fundamental symmetries in Krein C^* -algebras are remnants of the fundamental decompositions of the Krein spaces on which they are faithfully represented, their existence in situations coming from semi-Riemannian geometry seems to be a consequence of such global topological conditions of orientability and it is likely that a more general definition of a complete semi-definite analog of C^* -algebras might be necessary to deal with such cases. A possible line of attack would be to define semi-definite modules that are direct summand submodules of our “free-splitting” Krein modules over a C^* -algebra (eliminating the topological obstruction on orientability via “embedding” into a wider environment exactly as we usually do in the case of projective but non-free modules) and redefine Krein C^* -algebras as compressions of the “free-splitting” Kawamura case. We might explore these and other possibilities in subsequent work.

A more immediately achievable important goal (especially in view of applications to examples of semi-Riemannian geometries related to relativistic physics) is the removal of the unitality (compactness) requirements in the definitions of Krein C^* -modules and Krein C^* -algebras.

Our main long-term interest is to formulate notions of semi-definite involutive operator algebraic environments that are suitable, as a (topological) background, for the development of non-commutative geometry and spectral triples in a completely general semi-definite situation.

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